Advanced Statistics

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Problem Set 4. Probability inequalities, convergence

References

- FPP: Statistics (4/e), Freedman, Pisani, Purves
- Wasserman: All of Statistics, Springer, 2003
- JWHT: An Introduction to Statistical Learning with Applications in R, James, Witten, Hastie and Tibshirani, Springer, 2017
- VS: An Introduction to R, Venables and Smith

1. A coin is tossed 100 times. What is the probability of getting 90 or more heads? Compare the exact value against Markov and Chebyshev bounds.

2. State and prove the weak law of large numbers.

3. Let $X_1, X_2, \ldots, X_n$ be iid random variables where $X_i \sim \mathcal{U}([0,1])$. Find $\Pr(\sum_{i=1}^{20} X_i \geq 15)$ using Markov, Chebyshev and central limit theorem.

4. Prove that mean square convergence implies convergence in probability.

5. Let $X_n \sim \mathcal{N}(0, 1/n)$. Show that $X_n \xrightarrow{p} 0$.

6. Let the CDF $F_{X_n}(x) = 1 - (1 - 1/n)^{nx}$ when $x > 0$ and $F_{X_n}(x) = 0$, otherwise. Show that $X_n \xrightarrow{d} \mathcal{E}(1)$, where $\mathcal{E}$ denotes exponential distribution.

7. Let $X_n \sim \mathcal{G}(\lambda/n)$ where $\mathcal{G}$ denotes geometric distribution. If $Y_n = X_n/n$, show that $Y_n \xrightarrow{d} \mathcal{E}(\lambda)$.

8. Let $X_1, X_2, \ldots, X_n$ be iid random variables where $X_i \sim \text{Bern}(p)$. Define $S_n = X_1 + \ldots + X_n$. Find $\Pr(S_n \leq 30)$ where $n = 42, p = 0.5$. Find $\Pr(S_n = 30)$ using exact calculation and compare with the result produced using the central limit theorem.

9. Let $X \sim \mathcal{N}(25, 5)$. Find:

i. $\Pr(X > 30)$

ii. $\Pr(20 \leq X \leq 30)$

iii. $\Pr(|X - 30| > 10)$

10. Examination marks of a certain course is well approximated by $\mathcal{N}(70, 49)$. If 4 students are selected randomly, show that the the probability that exactly two of them attained at-least 84 marks is 0.003. What is the probability that exactly two of them have attained at-least 77?