Artificial Intelligence

17. Markov Decision Processes

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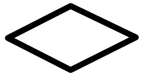
Recap: Decision Networks

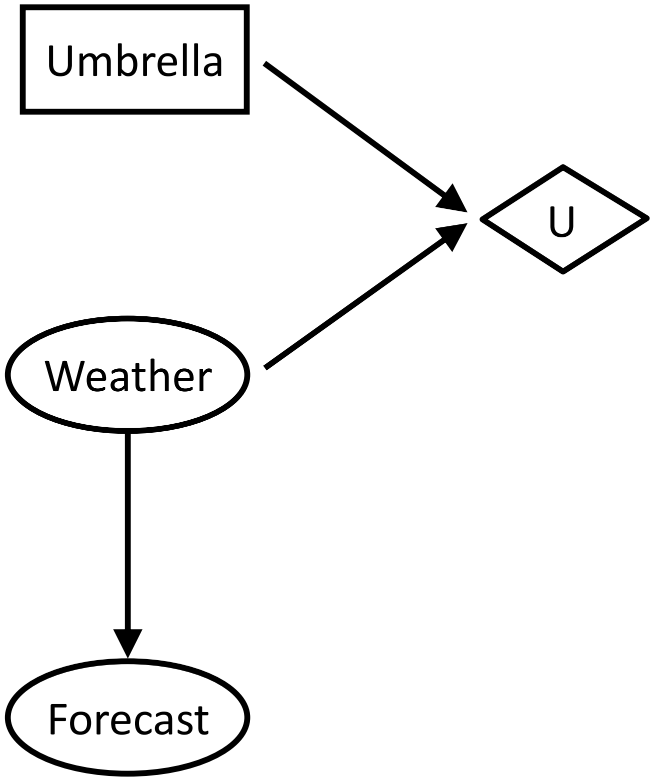
- Decision network = Bayes net + Actions + Utilities

- **Action nodes** (rectangles, cannot have parents, will have value fixed by algorithm)



- **Utility nodes** (diamond, depends on action and chance nodes)

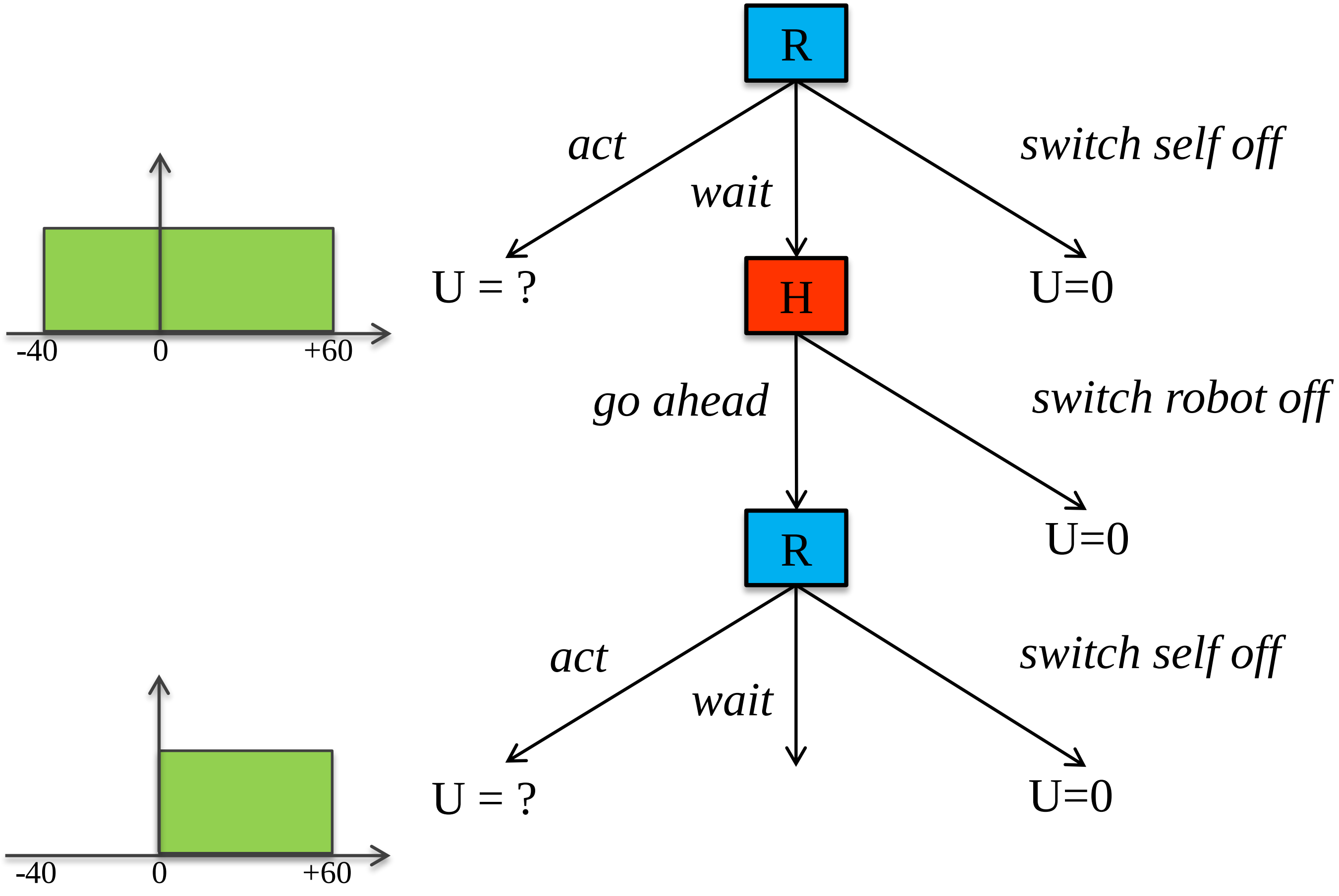
- Decision network represents a decision problem, containing all the information needed to for the agent to decide
  - What action to take given evidence ***e***
    - Decision algorithm
  - Value of information
    - $\mathrm{VPI}(E_i \mid e) = \left[\ \sum_{e_i} P(e_i \mid e) \max_a \mathrm{EU}(a \mid e_i, e)\ \right] - \max_a \mathrm{EU}(a \mid e)$



Decisions with unknown preferences

- In reality the assumption that we can write down our exact preferences for the machine to optimize is false
- A machine optimizing the wrong preferences causes problems
- A machine that is explicitly uncertain about the human's preferences will defer to the human (e.g., allow itself to be switched off)

Off-switch problem

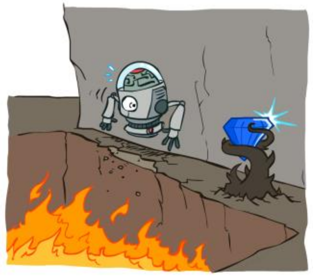


EU(act) = +10

EU(wait) = (0.4 * 0) + (0.6 * 30) = +18

Sequential decisions under uncertainty

- So far, decision problem was one-shot
    - Concerning only one action
- Sequential decision problem: agent's utility depends on a sequence of actions



Markov Decision Process (MDP)

Andrey Markov
(1856-1922)

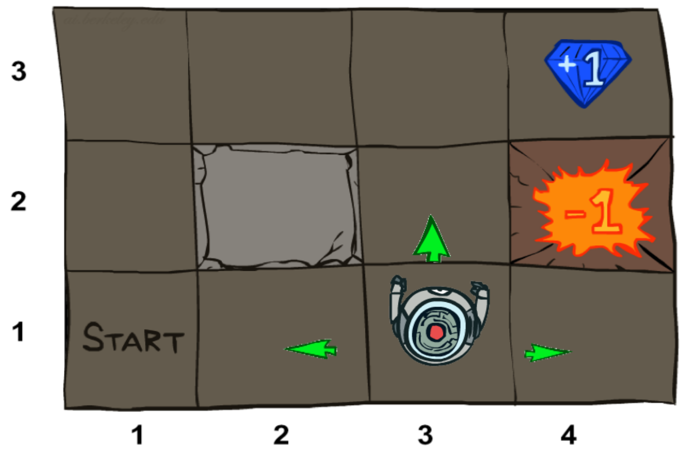
- Environment history: [s_0, a_0, s_1, a_1, …, s_t]
- **Markov** means that given the present state, the future and the past are independent : First Order Markov Chain
- For Markov decision processes, Markov means action outcomes depend only on the current state

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \ldots S_0 = s_0) = P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

- This is just like search, where the successor function could only depend on the current state (not the history)

Markov Decision Process (MDP)

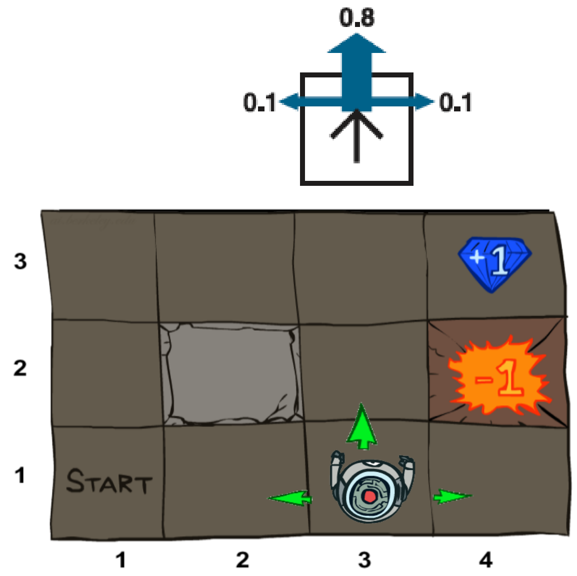
- An MDP is defined by:
  - A set of states $s \in S$
  - A set of actions $a \in A$
  - A transition model $T(s, a, s')$
    - Probability that a from s leads to s', i.e., $P(s' | s, a)$
  - A reward function $R(s, a, s')$ for each transition
  - A start state
  - Possibly a terminal state (or absorbing state)
  - Utility function which is additive (discounted) rewards



- MDPs are fully observable but probabilistic search problems

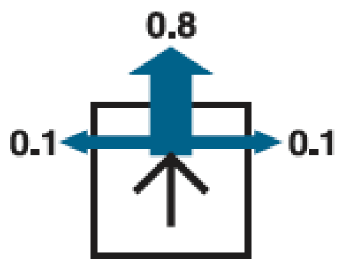
[Demo – gridworld]

Example: Grid World



- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put

Example: Grid World



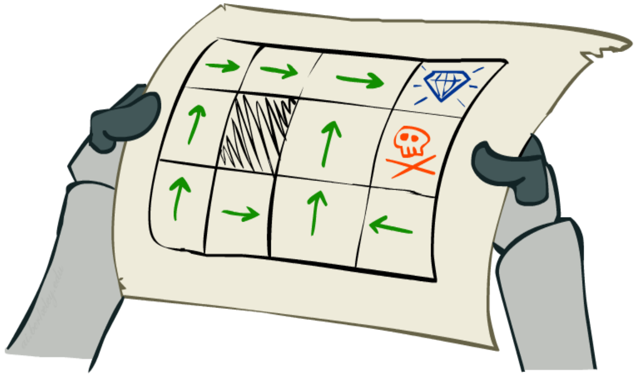
- The agent receives rewards each time step
  - Small “living” reward r each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards

Policies

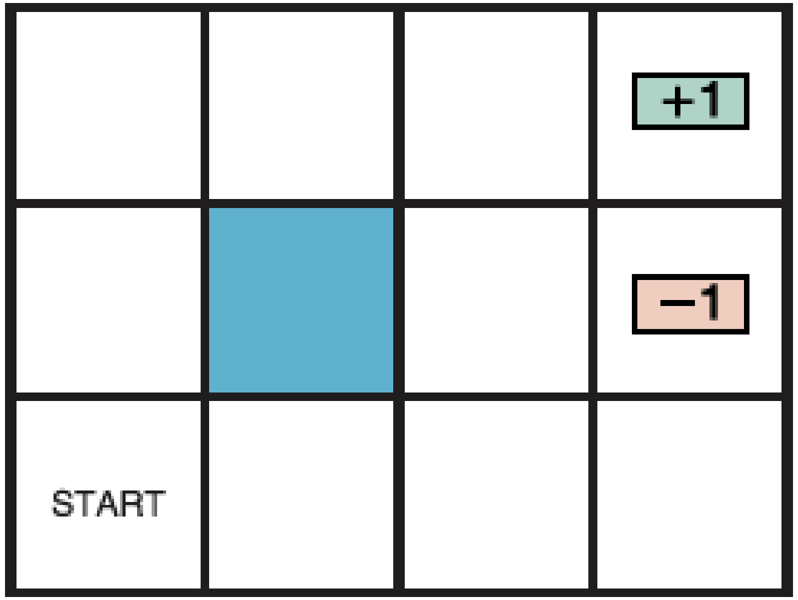
- A policy π gives an action for each state, $\pi: S \rightarrow A$

- In deterministic single-agent search problems, we wanted an optimal plan, or sequence of actions, from start to a goal

- For MDPs, we want an optimal policy $\pi^*: S \rightarrow A$
  - An optimal policy maximizes expected utility
  - An explicit policy defines a reflex agent

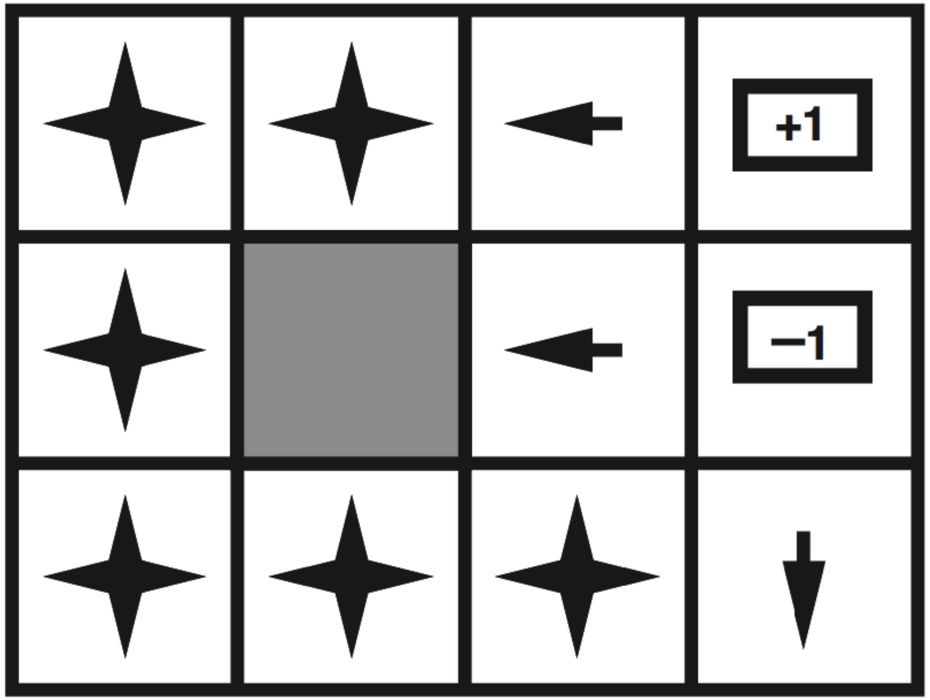


Optimal policy for r > 0



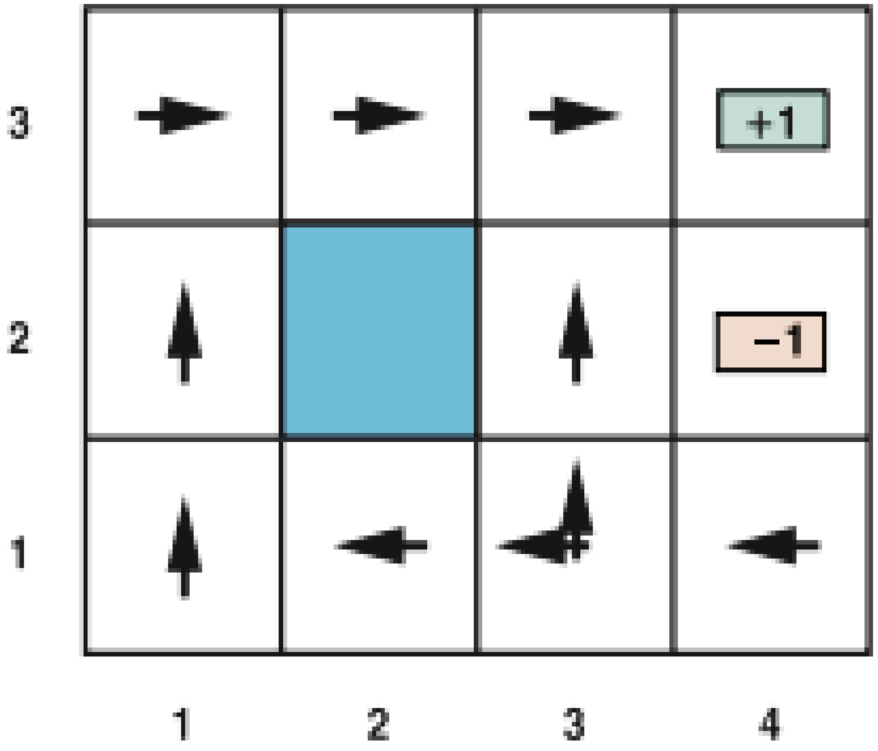
r > 0

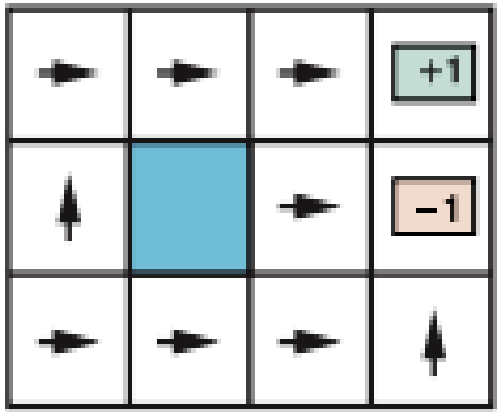
Optimal policy for r>0



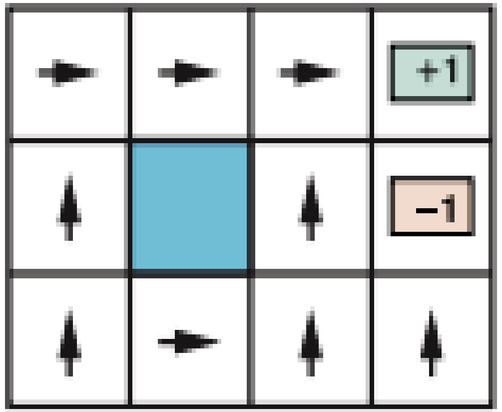
r > 0

Sample Optimal Policies

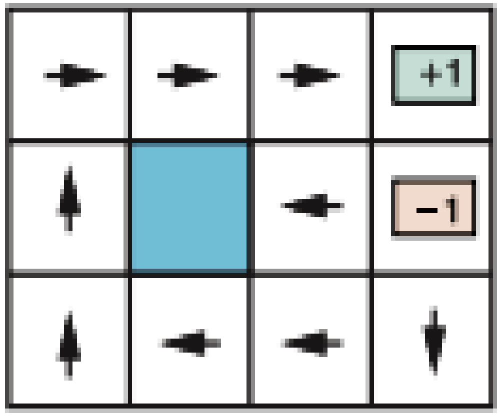




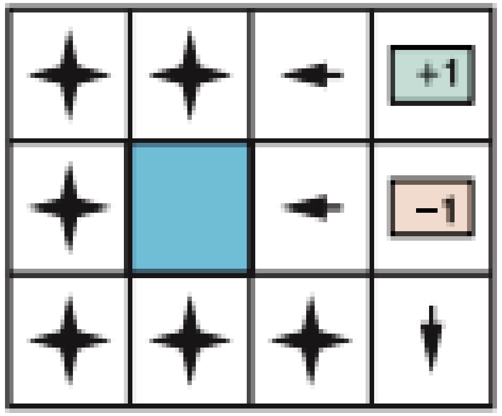
$r < -1.6497$



$-0.7311 < r < -0.4526$



$-0.0274 < r < 0$



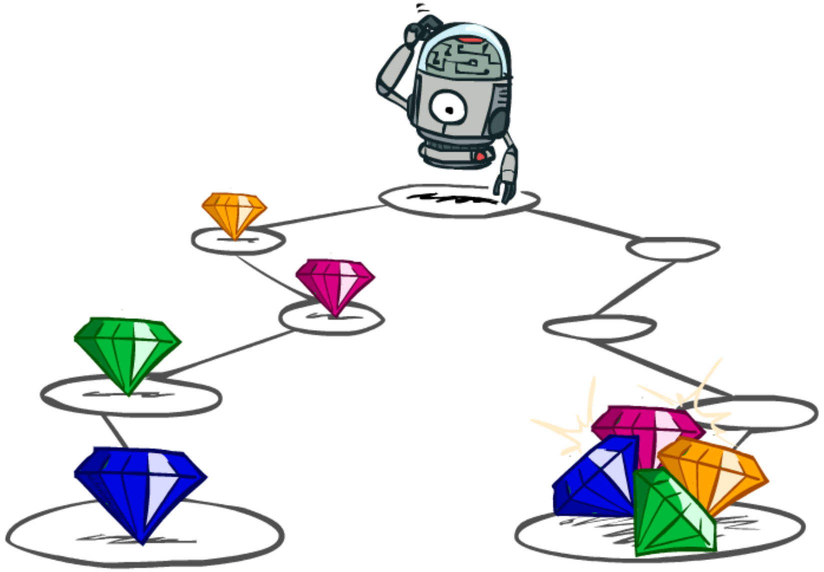
$r > 0$

Utilities of Sequences

- What preferences should an agent have over reward sequences?

- More or less? [1, 2, 2] or [2, 3, 4]

- Now or later? [0, 0, 1] or [1, 0, 0]



Discounting

- Discounting conveniently solves the problem of infinite reward streams!
  - Geometric series: $1 + \gamma + \gamma^2 + \ldots = 1/(1 - \gamma)$
  - Assume rewards bounded by $\pm\, R_{max}$
  - Then $r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots$ is bounded by $\pm\, R_{max}/(1 - \gamma)$

$$U_h([s_0, a_0, s_1, a_1, s_2, \ldots]) = R(s_0, a_0, s_1) + \gamma R(s_1, a_1, s_2) + \gamma^2 R(s_2, a_2, s_3) + \cdots$$

Worth r now

Worth γr next step

Worth $\gamma^2 r$ in two steps

Quiz: Discounting

- Given:

10				1
a	b	c	d	e

  - Actions: East, West, and Exit (only available in exit states a, e)
  - Transitions: deterministic

- Quiz 1: For $\gamma = 1$, what is the optimal policy?

10				1

- Quiz 2: For $\gamma = 0.1$, what is the optimal policy?

10				1

- Quiz 3: For which γ are West and East equally good when in state d?

The utility of a policy

- Executing a policy π from any state s_0 generates a sequence

$s_0, \pi(s_0), s_1, \pi(s_1), s_2, \ldots$

- This corresponds to a sequence of rewards

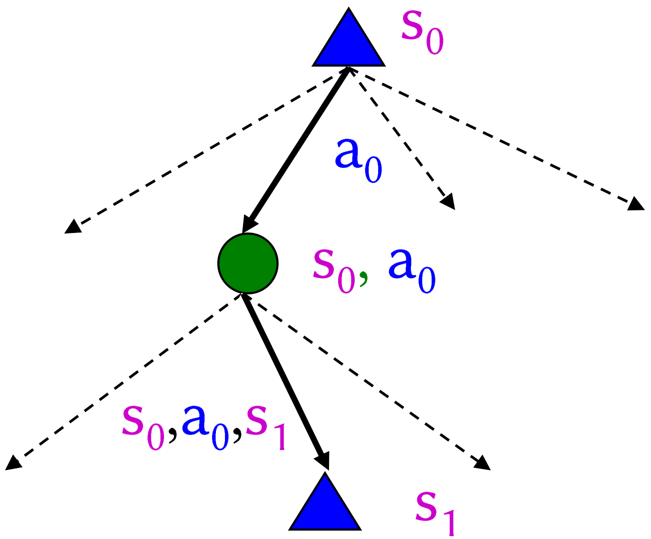
$R(s_0, \pi(s_0), s_1), R(s_1, \pi(s_1), s_2), \ldots$

- This reward sequence happens with probability

$P(s_1 \mid s_0, \pi(s_0)) \times P(s_2 \mid s_1, \pi(s_1)) \times \ldots$

- The value (expected utility) of π in s_0 is written $U^\pi(s_0)$
  - It's the sum over all possible state sequences of

    (discounted sum of rewards) x (probability of state sequence)



$$U^\pi(s_0) = E\left[\sum_{t=0}^{\infty} \gamma^t R(S_t, \pi(S_t), S_{t+1})\right]$$

Optimal Quantities

- The optimal policy:

  $\pi^*(s)$ = optimal action from state s

  Gives highest $U^{\pi}(s)$ for any π

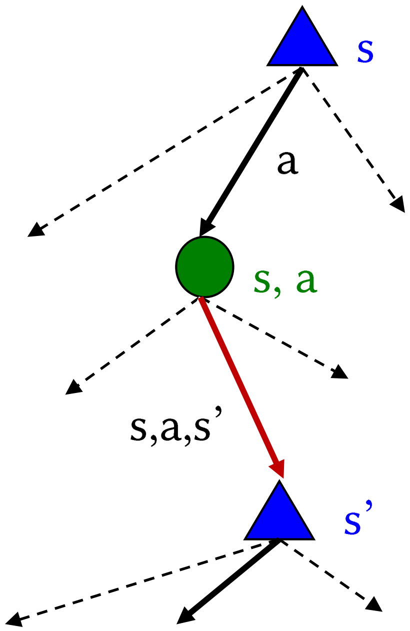
- The value (utility) of a state s:

  $U^*(s) = U^{\pi^*}(s)$ = expected utility starting in s and acting optimally

- The value (utility) of a q-state (s,a):

  $Q^*(s,a)$ = expected utility of taking action a in state s and (thereafter) acting optimally

  $U^*(s) = \max_a Q^*(s,a)$



s is a *state*

(s, a) is a *q-state*

(s,a,s') is a *transition*

Bellman equations (Shapley, 1953)

- The value/utility of a state is
    - The expected reward for the next transition plus the discounted value/utility of the next state, assuming the agent chooses the optimal action

- Hence we have a recursive definition of value (Bellman equation):

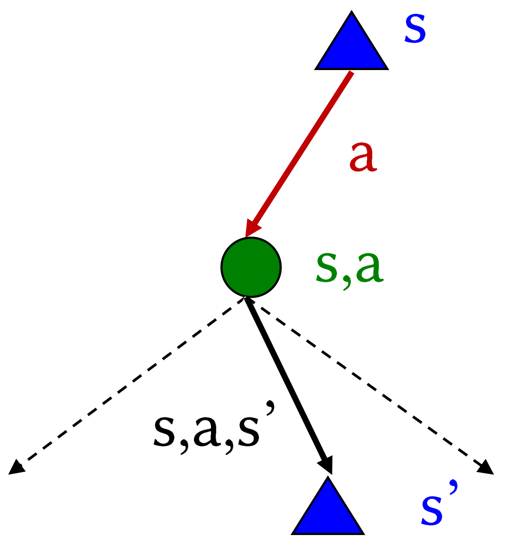
$$U(s) = \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a)[R(s, a, s') + \gamma U(s')]$$

- Similarly, Bellman equation for Q-functions

$$\begin{aligned} Q(s,a) &= \sum_{s'} P(s' \mid s, a)[R(s, a, s') + \gamma U(s')] \\ &= \sum_{s'} P(s' \mid s, a)[R(s, a, s') + \gamma \max_{a'} Q(s', a')] \end{aligned}$$

Value Iteration

- Start with (say) $U_0(s) = 0$ and some termination parameter ε
- Repeat until convergence (i.e., until all updates smaller than ε)
  - Do a **Bellman update** (essentially one ply of expectimax) from each state:
  - $U_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s' \mid a,s) \, [R(s,a,s') + \gamma U_k(s')]$



- Theorem: will converge to unique optimal values

Extracting policy

- How should the agent act given **U** (s)?
- Maximize expected utility! (as if **U** is correct)
- That is, do a mini-expectimax (greedy one-step):

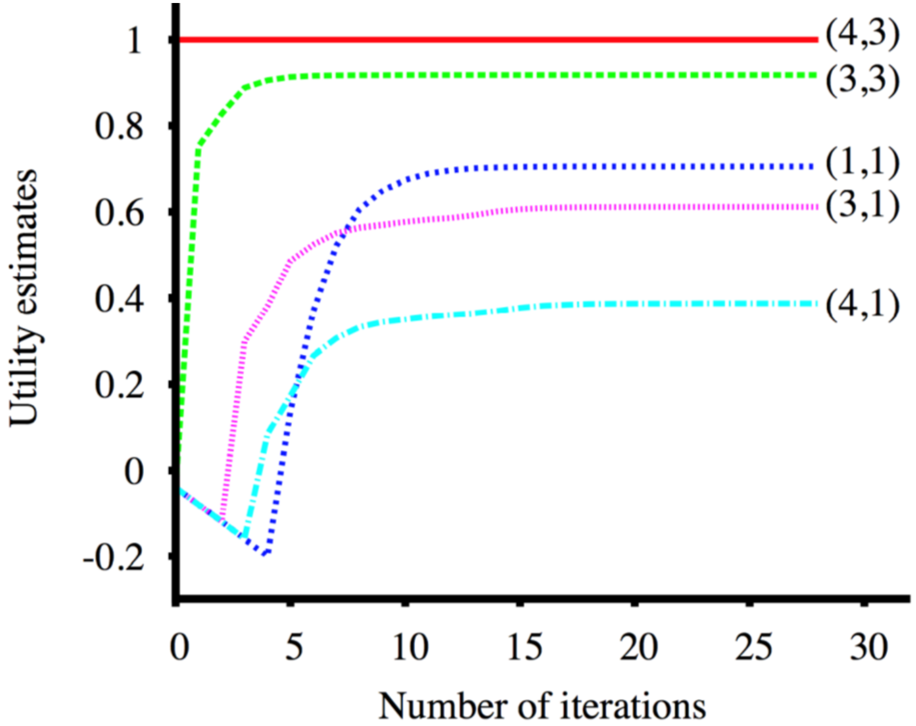
$$\pi_U(s) = \text{argmax}_a \sum_{s'} P(s' \mid a,s) [R(s,a,s') + \gamma \mathbf{U}(s')]$$

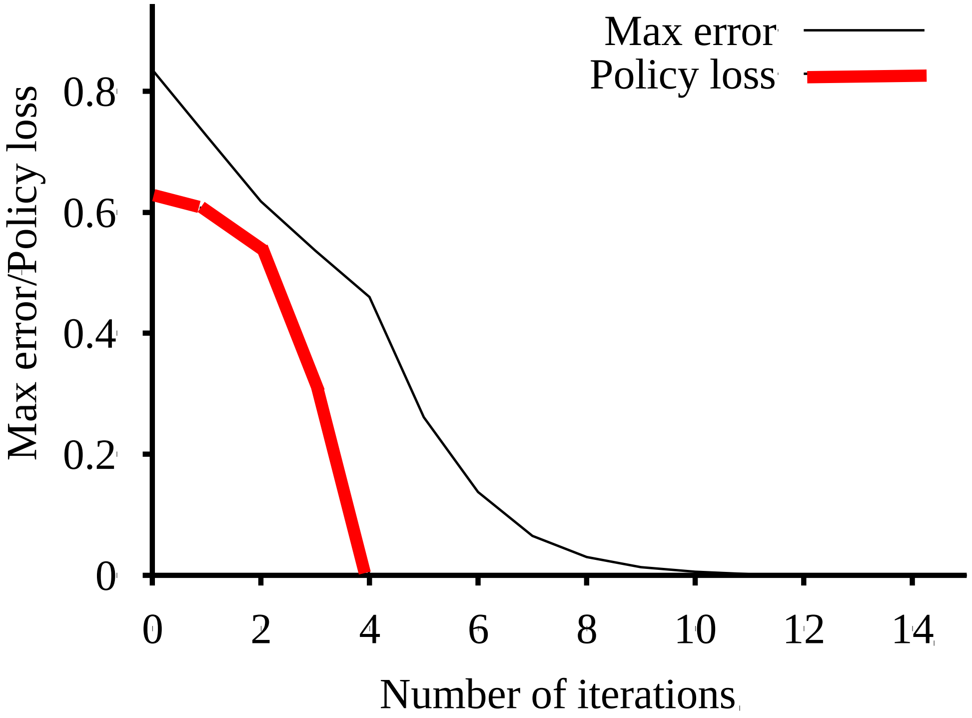
- This is called **policy extraction**, since it finds the policy π_U implied by the values **U**

How good is the policy extracted from VI?

- The quality of a policy π is measured by the policy loss $\| U^{\pi} - U^{*} \|$
- Let $\pi_k = \pi_{U_k}$ i.e. the implied policy at step k
  - When $\| U_k - U^* \| \leq \varepsilon$, policy loss is bounded:

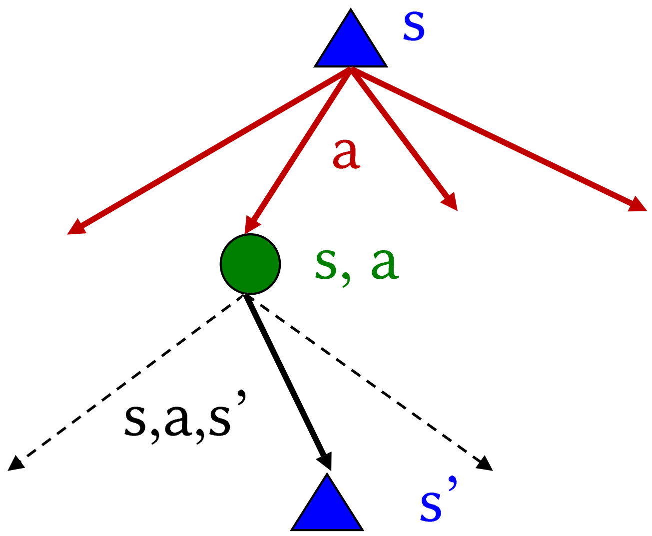
$$\| U^{\pi_k} - U^* \| \leq 2\varepsilon\gamma/(1-\gamma)$$



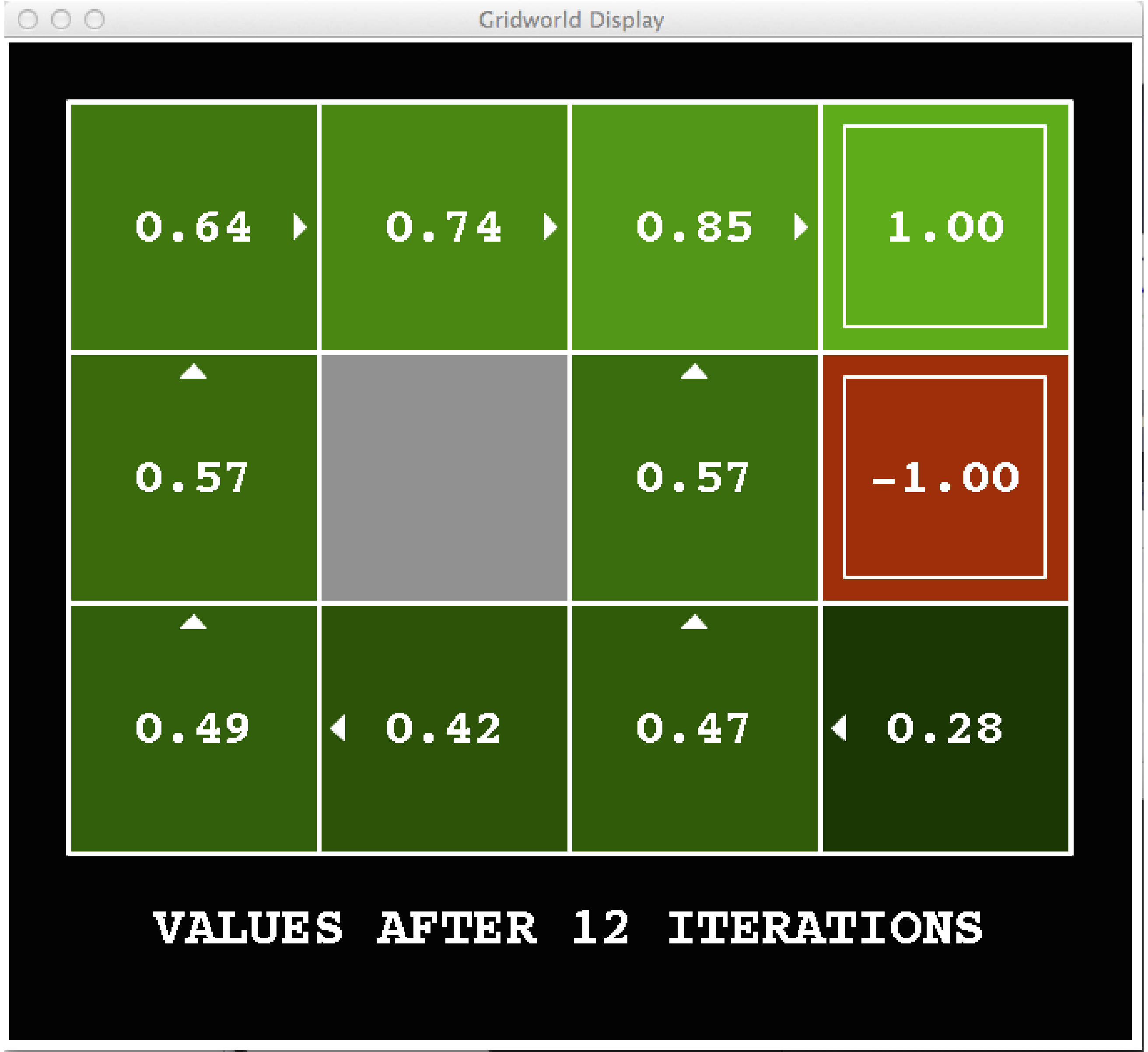


Problems with Value Iteration

- Value iteration repeats the Bellman updates:
  - $U_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s' \mid a,s)\,[R(s,a,s') + \gamma U_k(s')]$
- Problem 1: It's slow – $O(S^2A)$ per iteration
- Problem 2: The "max" at each state rarely changes
- Problem 3: The policy often converges long before the values

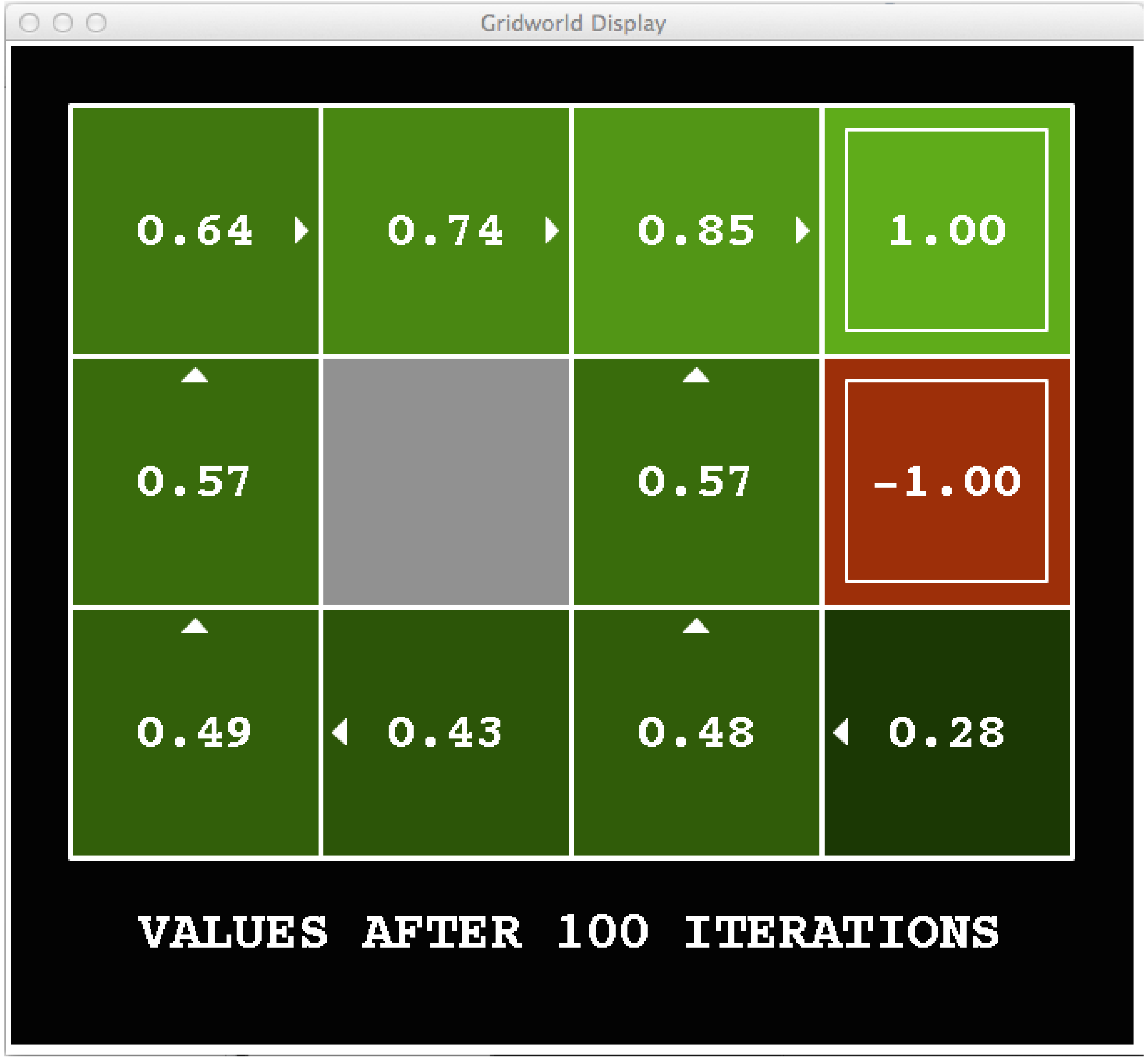


Policy Iteration



Noise = 0.2
Discount = 0.9
Living reward = 0

Policy Iteration



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Policy Iteration

- Basic idea: make the implied policy in U explicit, compute its long-term implications for value
- Repeat until no change in policy:
  - Step 1: Policy evaluation: calculate value U^{π_k} for current policy π_k
  - Step 2: Policy improvement: extract the new implied policy π_{k+1} from U^{π_k}

- It's still optimal!
- Can converge (much) faster under some conditions

Quiz

- Which of the following correctly defines a Markov Decision Process (MDP)?

  A. A deterministic search problem with a single terminal state
  B. A probabilistic model defined by states, actions, rewards, and transition probabilities
  C. A logical model defined by inference rules and operators
  D. A supervised learning task with labeled examples

- What does the Markov property state in an MDP?

  A. The future is independent of the past, given the present state.
  B. The future depends on all previous actions equally.
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- Which of the following expresses the Bellman optimality condition?
  A. $V(s)=R(s)$
  B. $V_*(s)=\max_a \Sigma_{s'} P(s'|s,a)[R(s,a,s')+\gamma V_*(s')]$
  C. $Q(s,a)=R(s,a)$
  D. $V_*(s)=\Sigma_a P(a|s)R(s,a)$
- How do value iteration and policy iteration differ?
  A. Value iteration alternates policy evaluation and improvement; policy iteration updates all values simultaneously.
  B. Policy iteration alternates between policy evaluation and improvement; value iteration merges them into a single update loop.
  C. Policy iteration uses exploration; value iteration does not.
  D. Value iteration is stochastic; policy iteration is deterministic.

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Quiz

- Why are MDPs central to modern AI?
  A. They describe one-shot classification problems.
  B. They model reasoning under certainty.
  C. They formalize sequential decision-making under uncertainty — the foundation of reinforcement learning.
  D. They only apply to deterministic planning tasks.

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Summary

- An MDP provides a mathematical framework for sequential decision making when outcomes are partly random and partly under the agent's control.
  - It formalizes how a rational agent should act to maximize long-term expected utility.
- Components of an MDP are States (S), Actions (A), Transition Model ($P(s' \mid s, a)$), Reward Function ($R(s, a, s')$) and Discount Factor ($0 \le \gamma < 1$).
- The Objective is to find a policy **π** that maps each state to the optimal action, maximizing the expected discounted sum of future rewards
- The Bellman Equations give the optimal value function satisfies the recursive relationship: $U^*(s) = \max_a \Sigma_{s'} P(s'|s,a)[R(s,a,s') + \gamma\, U^*(s')]$
  - The utility of a state equals its immediate reward plus the expected discounted value of the next state.
- Solving an MDP
  - Value Iteration: iteratively apply Bellman updates until values converge.
  - Policy Iteration: alternate between evaluating a policy and improving it.