Artificial Intelligence

14. Markov Models

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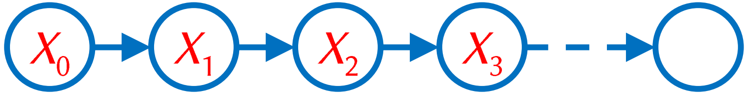
Uncertainty and Time

- Method: Belief state, transition model and sensor model
- Often, we want to reason about a sequence of observations where the state of the underlying system is changing
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
  - Global climate
- Now we need to introduce time into our models

Discrete-Time Model

- Discrete-time model views the problem as snapshots in time, called time slices
- Each time slice contains a set of random variables, some observable and some not
- We will assume the same subset of variables are observable in every time slice



Discrete-Time Model

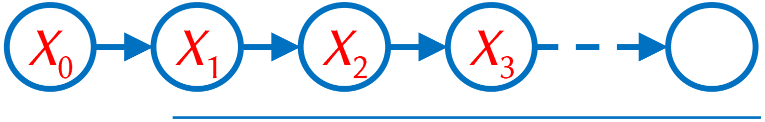
- Set of variables at time t:
  - X_t: state variables, assumed to be unobservable
  - E_t: evidence variables, set of observable variables
    - e_t: observations at time t
- Transition model $P(X_t \mid X_{0:t-1})$
  - Specifies the probability distribution over the latest state variables given the previous values
  - Problem: Set $X_{0:t-1}$ is unbounded in size

Notation:

$X_{a:b} = X_a , X_{a+1}, \ldots, X_b$

Markov Models

- **Markov assumption:** current state depends on only a finite fixed number of the past states
- Processes satisfying this assumption are called Markov chain or Markov processes
- First order Markov process -- the present state depends only on the previous state: $P(X_t \mid X_{0:t-1}) = P(X_t \mid X_{t-1})$
  - X_{t+1} is independent of $X_0, \ldots, X_{t-1}$ given X_t



$P(X_0)$ $\quad$ $\underline{\hspace{0pt}}$ $P(X_t \mid X_{0:t-1}) = P(X_t \mid X_{t-1})$

Markov Models

- Joint distribution $P(X_0,\ldots, X_T) = P(X_0) \prod_t P(X_t \mid X_{t-1})$
- A k^{th}-order model allows dependencies on k earlier steps
  - Higher order Markov chains can be transformed to the first order chain
- Time-homogeneous or Stationary process assumption
  - Not good we had to specify a different $P(X_t \mid X_{t-1})$ for each t
  - We assume that the transition probabilities are the same at all times
    - Usually a justifiable assumption
    - Helps to avoid specifying potentially infinite number of probabilities

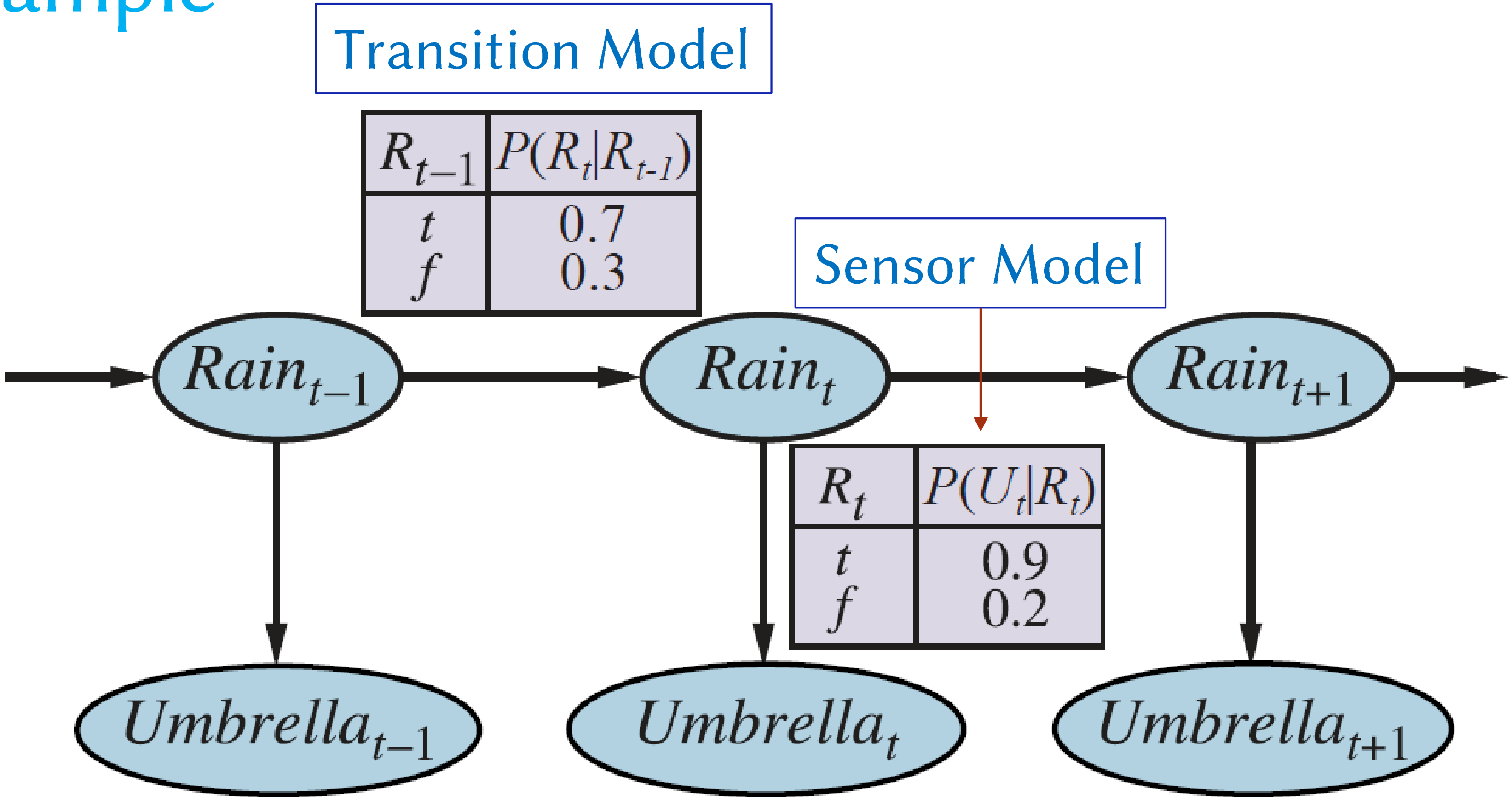
Markov Models

- Sensor Model (or, onservation model)
  - Evidence variable, $E_{t,}$ could depend on previous variables and observations leading to similar problem as the transition model
  - Sensor Markov assumption: a current state suffices to generate the current sensor values

$$P(E_t \mid X_{0:t}, E_{1:t-1}) = P(E_t \mid X_t)$$

  - $P(E_t \mid X_t)$ is our sensor model

Example



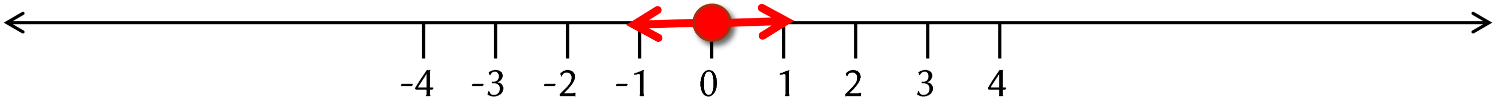
Transition Model

R_{t-1}	$P(R_t \mid R_{t-1})$
t	0.7
f	0.3

Sensor Model

R_t	$P(U_t \mid R_t)$
t	0.9
f	0.2

Example: Random walk in one dimension



- State: location on the unbounded integer line
- Initial state: starts at 0
- Transition model: $P(X_t = k | X_{t-1} = k \pm 1) = 0.5$
- Applications: particle motion in crystals, stock prices, gambling, genetics, …
- How far does it get as a function of t?
  - Expected distance is $O(\sqrt{t})$
- Does it get back to 0 or can it go off for ever and not come back?
  - In 1D and 2D, returns w.p. 1; in 3D, returns w.p. 0.34053733

Example: Weather

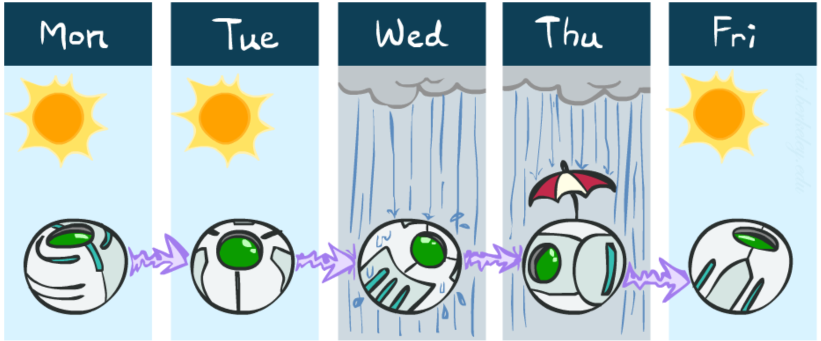
- States {rain, sun}

- Initial distribution $P(X_0)$

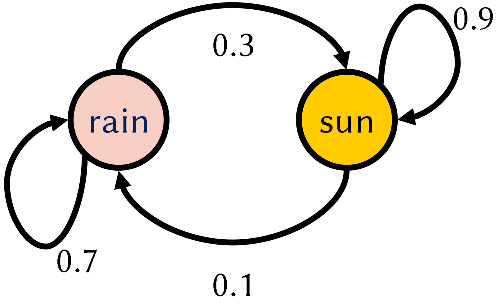
$P(X_0)$	
sun	rain
0.5	0.5

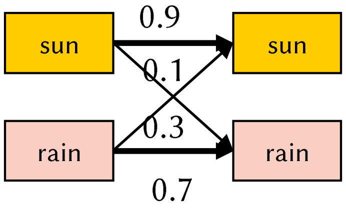
- Transition model $P(X_t \mid X_{t-1})$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



More ways of representing the same CPT



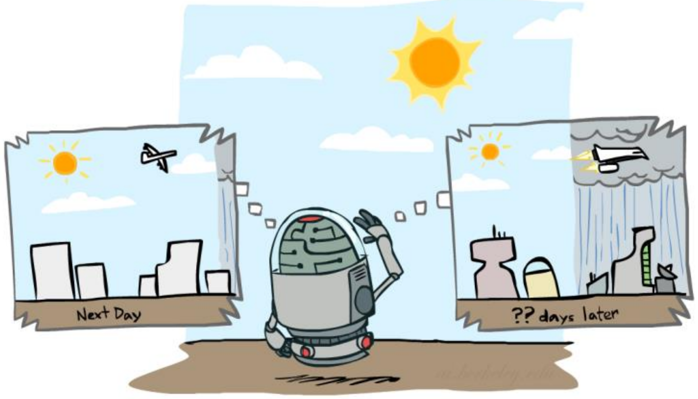


Weather prediction

- Time 0: <0.5, 0.5>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- What is the weather like at time 1?

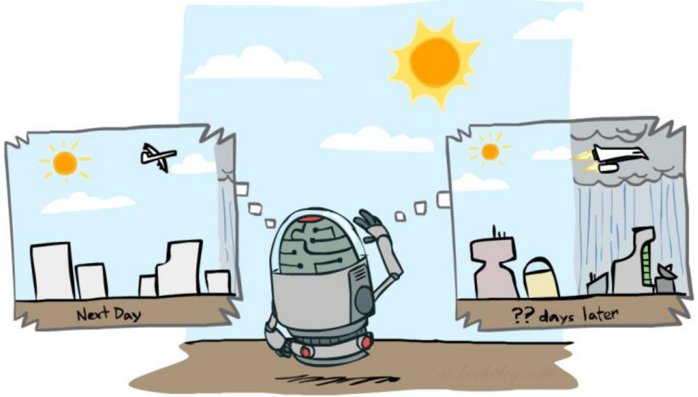


$$
\begin{aligned}
P(X_1) &= \textstyle\sum_{x_0} P(X_1, X_0{=}x_0) \\
&= \textstyle\sum_{x_0} P(X_0{=}x_0)\, P(X_1 \mid X_0{=}x_0) \\
&= 0.5\langle 0.9, 0.1\rangle + 0.5\langle 0.3, 0.7\rangle = \langle 0.6, 0.4\rangle
\end{aligned}
$$

Weather prediction, contd.

- Time 1: <0.6, 0.4>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7



- What is the weather like at time 2?

$$P(X_2) = \sum_{x_1} P(X_2, X_1=x_1)$$

$$= \sum_{x_1} P(X_1=x_1)\, P(X_2 | X_1=x_1)$$

$$= 0.6<0.9, 0.1> + 0.4<0.3, 0.7> = <0.66, 0.34>$$

Weather prediction, contd.

- Time 2: <0.66, 0.34>

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

- What is the weather like at time 3?

Homework

$P(X_0)$	
sun	rain
0	1

$$P(X_3) = \sum_{x_2} P(X_3, X_2 = x_2)$$

$$= \sum_{x_2} P(X_2 = x_2)\, P(X_3 \mid X_2 = x_2)$$

$$= 0.66<0.9,\ 0.1> + 0.34<0.3,\ 0.7> = <0.696,\ 0.304>$$

- The influence of initial distribution gets less and less over time. The distribution much later becomes independent of the initial distribution

Forward algorithm (simple form)

Transition model

- What is the state at time t?
  - $P(X_t) = \sum_{x_{t-1}} P(X_t, X_{t-1}=x_{t-1})$
  - $= \sum_{x_{t-1}} P(X_{t-1}=x_{t-1}) \, P(X_t | X_{t-1}=x_{t-1})$
- Iterate this update starting at $t=0$
  - This is called a **recursive** update:

$$P_t = g(P_{t-1}) = g(g(g(g(\ldots P_0))))$$

And the same thing in linear algebra

- What is the weather like at time 2?

  $P(X_2) = 0.6 <0.9,0.1> + 0.4 <0.3,0.7> = <0.66,0.34>$

- In matrix-vector form:

$$P(X_2) = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.66 \\ 0.34 \end{pmatrix}$$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

i.e., multiply by T^T, transpose of transition matrix

Stationary Distributions

- The limiting distribution is called the **stationary distribution** P_∞ of the chain
- It satisfies $P_\infty = P_{\infty+1} = T^T P_\infty$
- Solving for P_∞ in the example:

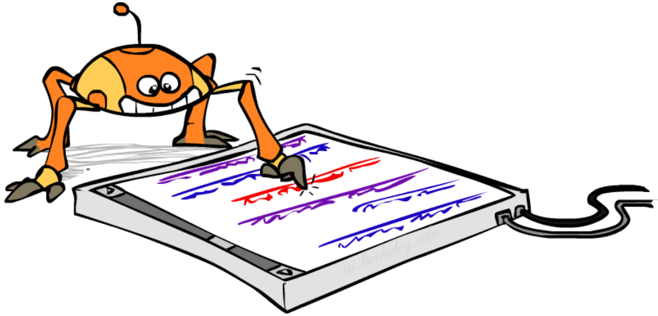
$$\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

$0.9p + 0.3(1-p) = p$, or, $p = 0.75$

- Stationary distribution is <0.75,0.25> **regardless of the starting distribution**

Example: Web browsing

- State: URL visited at step t
- Transition model:
  - With probability p, choose an outgoing link at random
  - With probability (1-p), choose an arbitrary new page
- Question: What is the **stationary distribution** over pages?
  - That is, if the process runs forever, what fraction of time does it spend in any given page?
- Application: Webpage ranking



Quiz

- Which statement correctly captures the **Markov assumption** used in a Hidden Markov Model?
  A. The current evidence depends only on the previous evidence.
  B. The current state depends on all previous states.
  C. The current state depends only on the previous state.
  D. The evidence is conditionally independent of the current state.

Quiz

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Quiz

1. Why can’t a static Bayesian network adequately model an agent’s beliefs about a changing world?

2. In a temporal model, what are the two main assumptions that simplify inference?

Quiz

1. Why can't a static Bayesian network adequately model an agent's beliefs about a changing world?

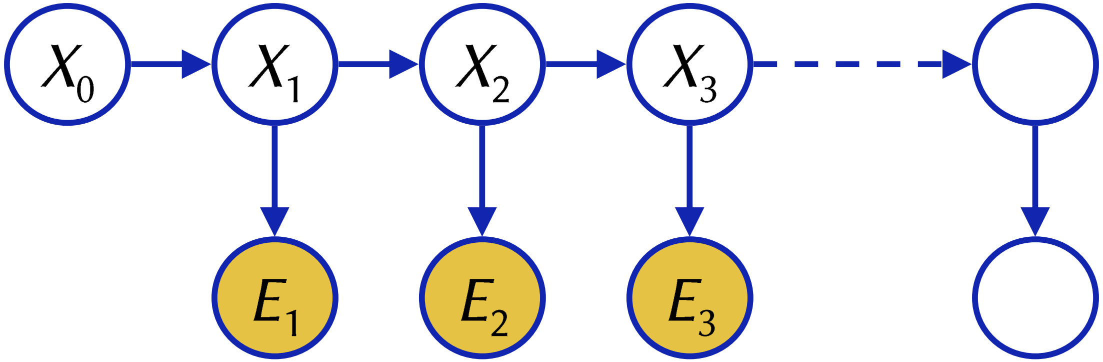
   - BN assumes that variables are independent of time. In dynamic environments, the current state depends on previous states, requiring temporal dependence be captured. Dynamic Bayesian Networks (DBNs) or Hidden Markov Models (HMMs) do just that.

2. In a temporal model, what are the two main assumptions that simplify inference?

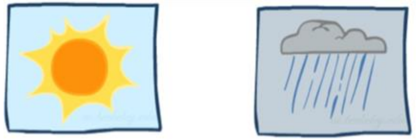
   - Markov assumption: $P(X_t|X_{1:t-1})=P(X_t|X_{t-1})$
   - Stationary process assumption: transition and sensor models do not change over time, i.e., $P(X_t|X_{t-1})$ and $P(E_t|X_t)$ are constant.

Hidden Markov Models (HMM)

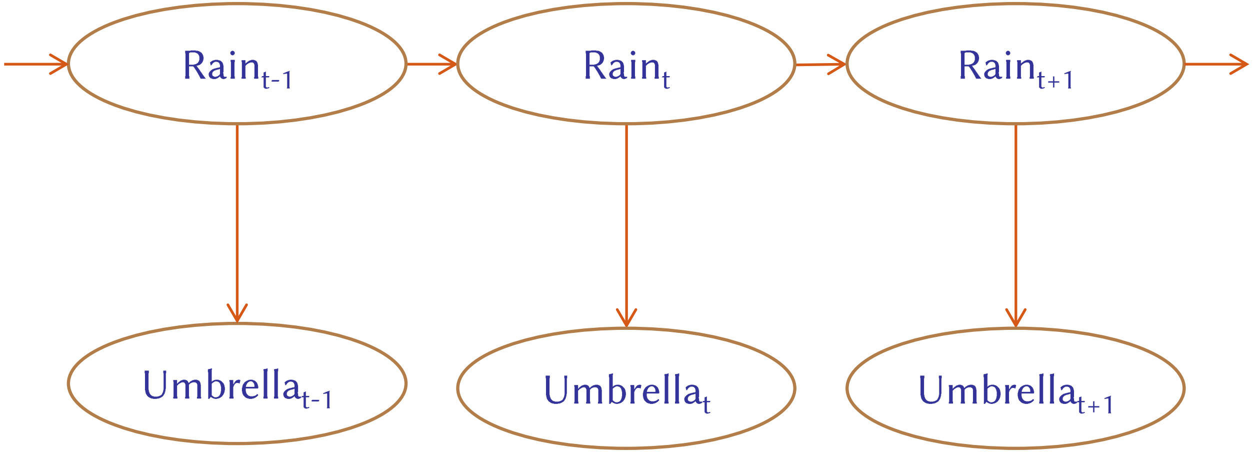
- Underlying Markov chain over belief states X
- You observe evidence E at each time step



Example: Weather HMM



- An HMM is defined by
  - Initial distribution: $P(X_0)$
  - Transition model: $P(X_t| X_{t-1})$
  - Sensor model: $P(E_t | X_t)$



R_{t-1}	$P(R_t \| R_{t-1})$	
	rain	¬rain
t	0.7	0.3
f	0.3	0.7

R_t	$P(U_t \| R_t)$	
	true	false
t	0.9	0.1
f	0.2	0.8

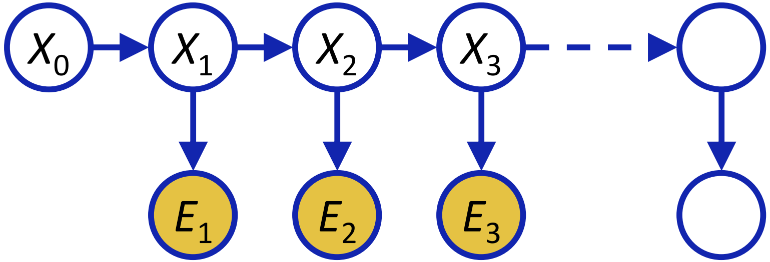
HMM as probability model

- Joint distribution for MM: $P(X_{0:t}) = P(X_0) \prod_{i=1:t} P(X_i \mid X_{i-1})$
- Joint distribution for HMM:

$$P(X_{0:t}, E_{1:t}) = P(X_0) \prod_{i=1:t} P(X_i \mid X_{i-1}) P(E_i \mid X_i)$$

  - Future states are independent of the past given the present
  - Current evidence is independent of everything else given the current state

- Are evidence variables independent of each other?



Useful notation:

$X_{a:b} = X_a , X_{a+1}, \ldots, X_b$

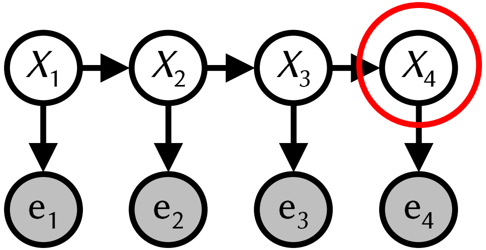
Real HMM Examples

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

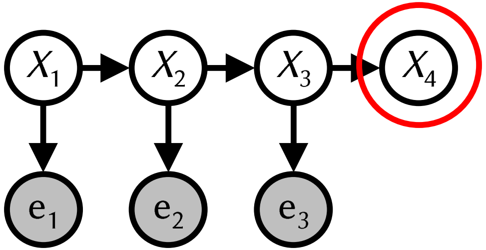
Inference tasks

- Filtering: $P(X_t|e_{1:t})$
  - Find belief state given evidences —input to the decision process of a rational agent, signal processing origin of the term
- Prediction: $P(X_{t+k}|e_{1:t})$ for $k > 0$
  - Evaluation of possible action sequences; like filtering without the evidence
- Smoothing: $P(X_k|e_{1:t})$ for $0 \leq k < t$
  - Better estimate of past states, essential for learning
- Most likely explanation: $\arg\max_{x_{1:t}} P(x_{1:t} \mid e_{1:t})$
  - Speech recognition, decoding with a noisy channel

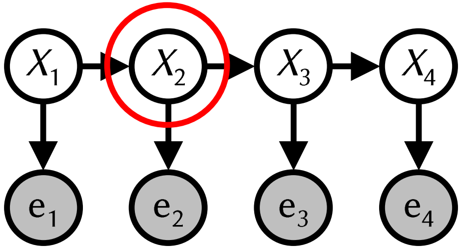
Filtering: $P(X_t|e_{1:t})$



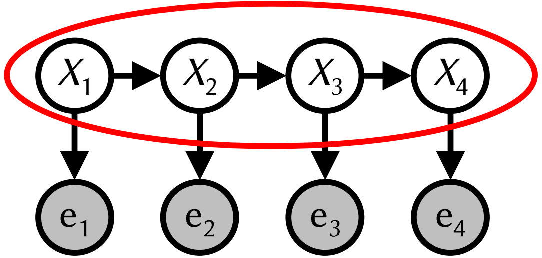
Prediction: $P(X_{t+k}|e_{1:t})$



Smoothing: $P(X_k|e_{1:t})$, $k<t$



Explanation: $P(X_{1:t}|e_{1:t})$



Filtering / Monitoring

- Filtering, or monitoring, or state estimation, is the task of maintaining the distribution $\mathbf{f}_{1:t} = P(X_t|e_{1:t})$ over time
- We start with $\mathbf{f}_0$ in an initial setting, usually uniform
- Filtering is a fundamental task in engineering and science
- The Kalman filter (continuous variables, linear dynamics, Gaussian noise) was invented in 1960 and used for trajectory estimation in the Apollo program
  - Core ideas used by Gauss for planetary observations
  - 13,80,000 references on Google Scholar [Nov 2025]

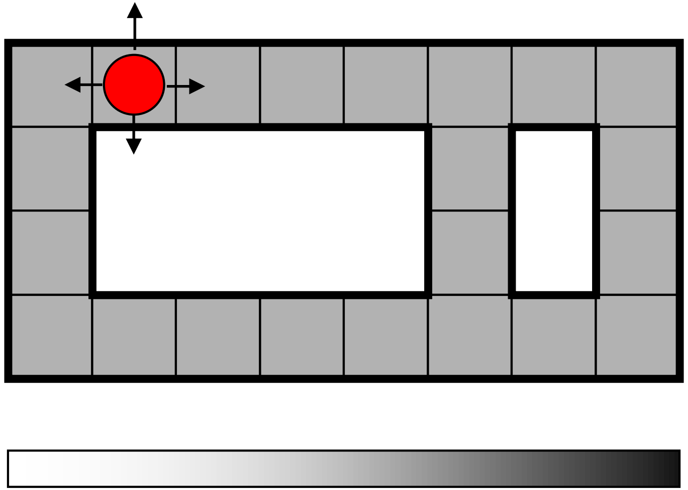
Example: Robot Localization

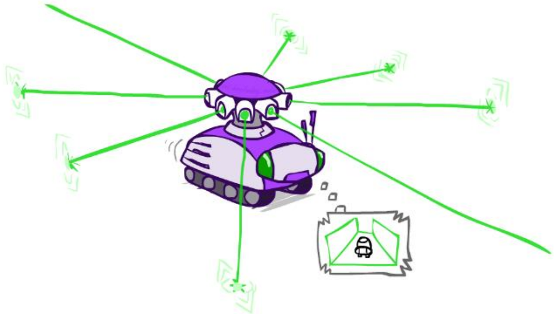
t=0

Sensor model: four bits for wall/no-wall in each direction, never more than 1 mistake

Transition model: action may fail with small probability

Example from Michael Pfeiffer

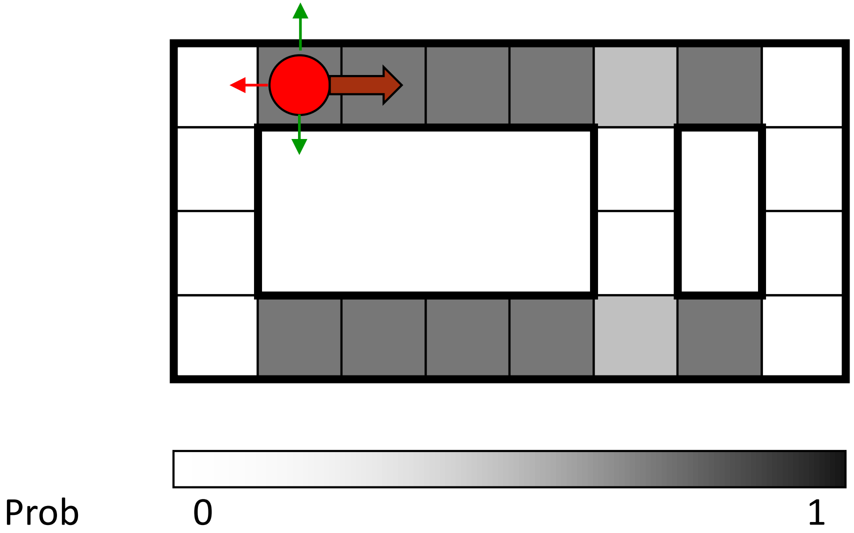


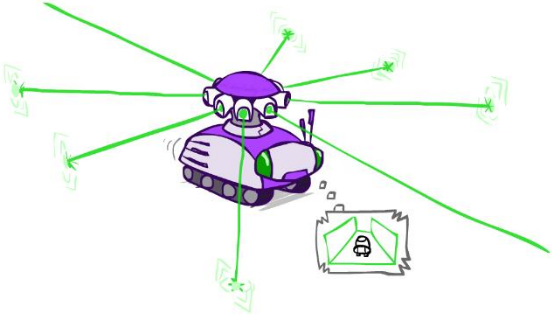


Example: Robot Localization

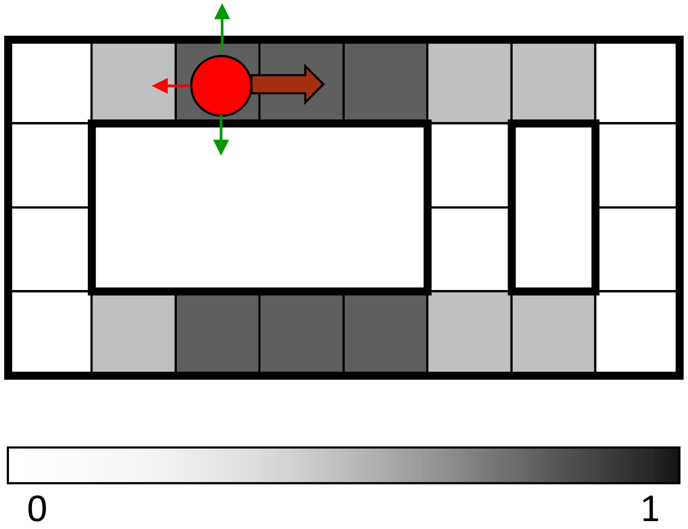
t=1

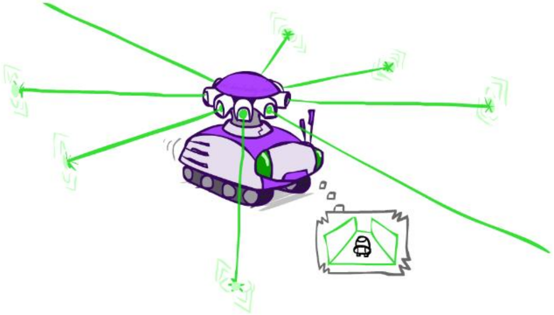
Lighter grey: was *possible* to get the reading, but *less likely* (required 1 mistake)





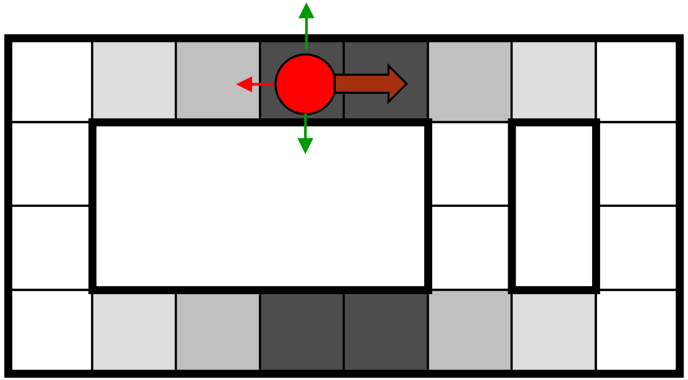
Example: Robot Localization

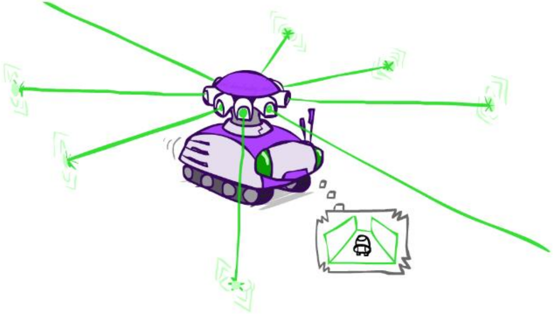




Example: Robot Localization

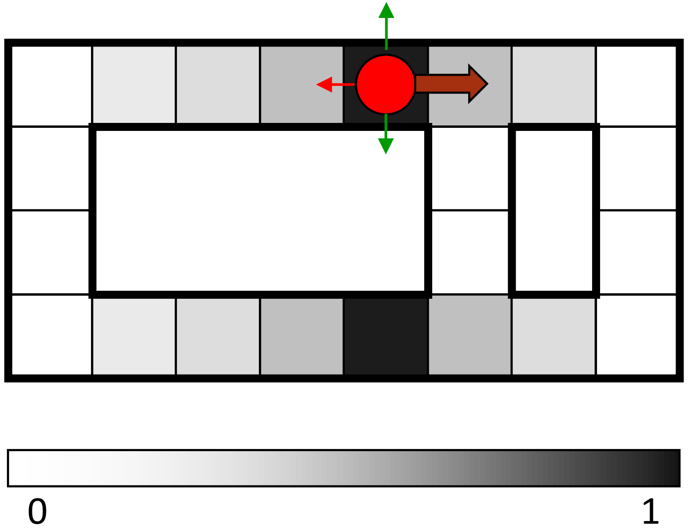
t=3

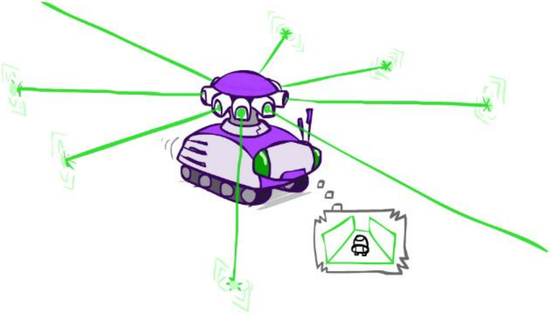




Example: Robot Localization

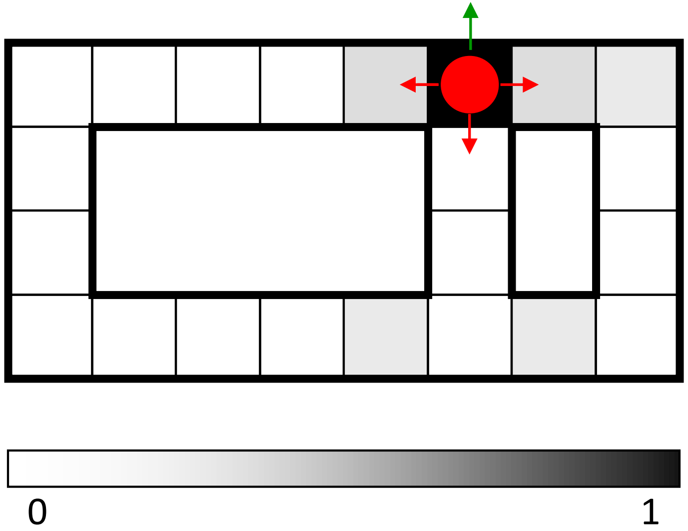
t=4

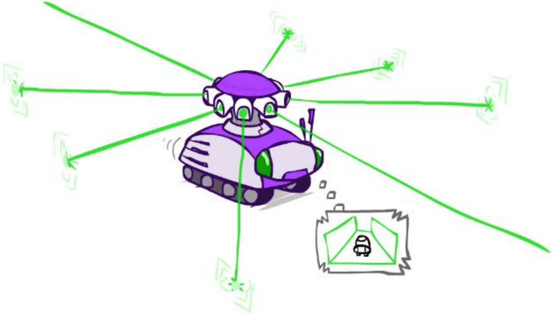




Example: Robot Localization

t=5





Filtering algorithm

- Aim: devise a **recursive filtering** algorithm of the form
  - $P(X_{t+1}|e_{1:t+1}) = g(e_{t+1}, P(X_t|e_{1:t}))$

- $P(X_{t+1}|e_{1:t+1}) =$

Filtering algorithm

- Aim: devise a **recursive filtering** algorithm of the form
  - $P(X_{t+1}|e_{1:t+1}) = f(e_{t+1}, P(X_t|e_{1:t}))$

- $P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}|e_{1:t}, e_{t+1})$

Filtering algorithm

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- $P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}|e_{1:t}, e_{t+1})$

$$= \alpha\, P(e_{t+1}|X_{t+1}, e_{1:t})\, P(X_{t+1}| e_{1:t})$$

$$= \alpha\, \underline{P(e_{t+1}|X_{t+1})}\, P(X_{t+1}| e_{1:t})$$

Apply sensor Markov conditional independence

Filtering algorithm

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$$= \alpha\, P(e_{t+1}|X_{t+1}, e_{1:t})\, P(X_{t+1}| e_{1:t})$$

$$= \alpha\, P(e_{t+1}|X_{t+1})\, P(X_{t+1}| e_{1:t})$$

$$= \alpha\, P(e_{t+1}|X_{t+1})\, \underline{\sum_{x_t}}\ \underline{P(x_t \mid e_{1:t})}\ \underline{P(X_{t+1}| x_t, e_{1:t})}$$

Condition on X_t

Filtering algorithm

- Aim: devise a **recursive filtering** algorithm of the form
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- $P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}|e_{1:t}, e_{t+1})$

$$= \alpha\, P(e_{t+1}|X_{t+1}, e_{1:t})\, P(X_{t+1}|\, e_{1:t})$$

$$= \alpha\, P(e_{t+1}|X_{t+1})\, P(X_{t+1}|\, e_{1:t})$$

Apply conditional independence

$$= \alpha\, P(e_{t+1}|X_{t+1}) \textstyle\sum_{x_t} P(x_t \mid e_{1:t})\, P(X_{t+1}|\, x_t, e_{1:t})$$

$$= \underbrace{\alpha}_{\text{Normalize}}\, \underbrace{P(e_{t+1}|X_{t+1})}_{\text{Sensor model}} \textstyle\sum_{x_t} \underbrace{P(X_{t+1}|\, x_t)}_{\text{Transition model}}\ \underbrace{P(x_t \mid e_{1:t})}_{\text{Recursion}}$$

Filtering algorithm

- $P(X_{t+1}|e_{1:t+1}) = \underbrace{\alpha}_{\text{Normalize}} \underbrace{P(e_{t+1}|X_{t+1})}_{\text{Update}} \underbrace{\sum_{x_t} P(x_t \mid e_{1:t}) P(X_{t+1} \mid x_t)}_{\text{Predict}}$

- $\mathbf{f}_{1:t+1} = \text{F}\textsc{orward}(\mathbf{f}_{1:t} , e_{t+1})$
- Cost per time step: $O(|X|^2)$ where $|X|$ is the number of states

- Time and space costs are **constant**, independent of t
- $O(|X|^2)$ is infeasible for models with many state variables

- We get to invent really cool approximate filtering algorithms

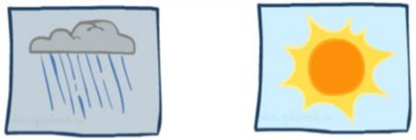
And the same thing in linear algebra

- Transition matrix T, observation matrix O_t
  - Observation matrix has state likelihoods for E_t along diagonal
  - E.g., for U_1 = true, $O_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.9 \end{pmatrix}$
- Filtering algorithm becomes
  - $\mathbf{f}_{1:t+1} = \alpha\, O_{t+1} T^{T}\, \mathbf{f}_{1:t}$

X_{t-1}	$P(X_t \mid X_{t-1})$	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

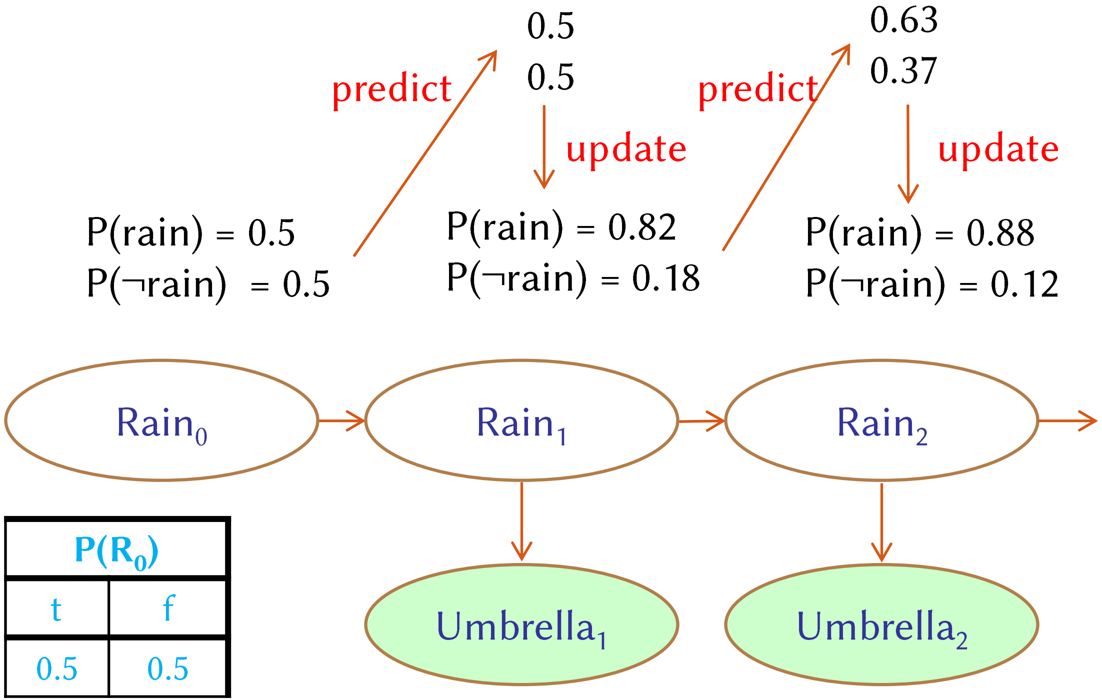
W_t	$P(U_t \mid W_t)$	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Example: Rain HMM



R_{t-1}	$P(R_t \mid R_{t-1})$	
	t	f
t	0.7	0.3
f	0.3	0.7

R_t	$P(U_t \mid R_t)$	
	true	false
t	0.9	0.1
f	0.2	0.8



predict

0.5
0.5

update

predict

0.63
0.37

update

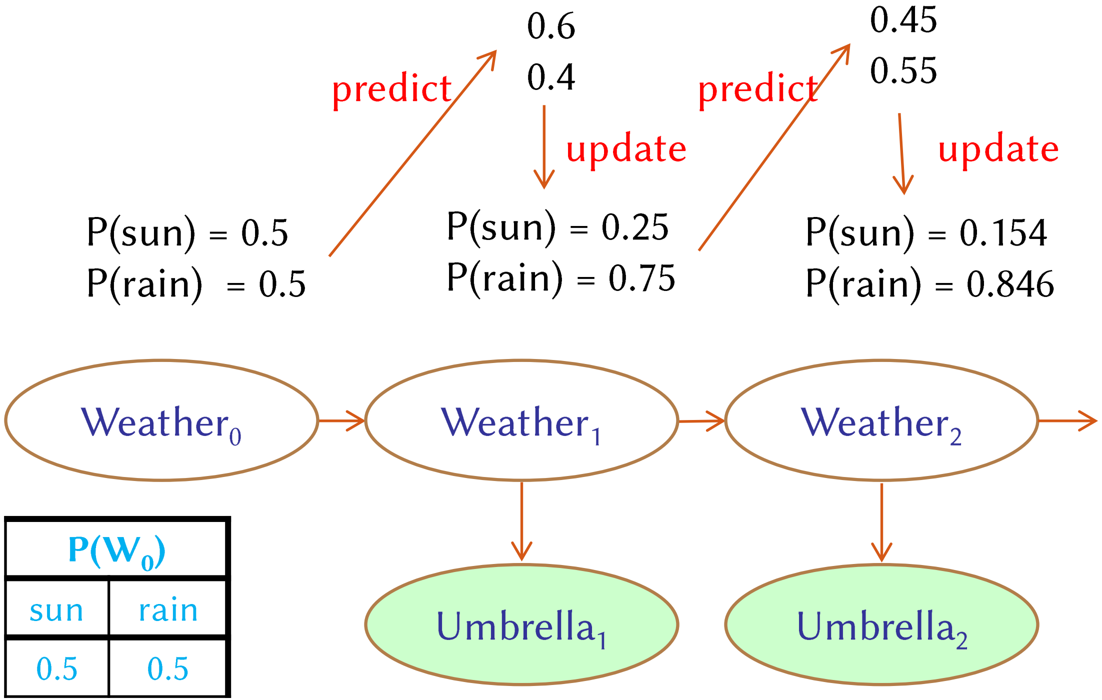
P(rain) = 0.5
P(¬rain) = 0.5

P(rain) = 0.82
P(¬rain) = 0.18

P(rain) = 0.88
P(¬rain) = 0.12

$P(R_0)$	
t	f
0.5	0.5

Example 2: Weather HMM



predict

0.6
0.4

update

P(sun) = 0.5
P(rain) = 0.5

P(sun) = 0.25
P(rain) = 0.75

predict

0.45
0.55

update

P(sun) = 0.154
P(rain) = 0.846

P(W_0)	
sun	rain
0.5	0.5

W_{t-1}	**P(W_t|W_{t-1})**	
	sun	rain
sun	0.9	0.1
rain	0.3	0.7

W_t	**P(U_t|W_t)**	
	true	false
sun	0.2	0.8
rain	0.9	0.1

Quiz

1. Which of the following best describes the goal of filtering in temporal probabilistic reasoning?

   A. Estimating the most likely future state of the system.

   B. Estimating the distribution over the current state given all past evidence.

   C. Estimating the probability of all previous states given current evidence.

2. In the recursive filtering equation, $P(X_{t+1}|e_{1:t+1}) = \alpha P(e_{t+1}|X_{t+1}) \sum_{x_t} P(x_t | e_{1:t}) P(X_{t+1} | x_t)$, what does the summation over x_t represent?

   A. Normalization to ensure probabilities sum to 1

   B. Marginalization over all possible previous states

   C. Evidence update using the sensor model

Quiz

1. Which of the following best describes the goal of filtering in temporal probabilistic reasoning?

A. Estimating the most likely future state of the system.

B. Estimating the distribution over the current state given all past evidence.

C. Estimating the probability of all previous states given current evidence.

2. In the recursive filtering equation, $P(X_{t+1}|e_{1:t+1}) = \alpha\ P(e_{t+1}|X_{t+1}) \sum_{x_t} P(x_t \mid e_{1:t})\ P(X_{t+1} \mid x_t)$, what does the summation over x_t represent?

A. Normalization to ensure probabilities sum to 1

B. Marginalization over all possible previous states

C. Evidence update using the sensor model

- **Reading**: Chapter 14
- **Assignments**: PS 8

- **Next**: Chapter 16. Utility theory